

On the lower bound of the discrepancy of Halton's sequence

Mordechay B. Levin^a,

^a*Department of Mathematics, Bar-Ilan University, Ramat-Gan, 52900, Israel*

Reçu le ***** ; accepté après révision le ++++++

Présenté par

Abstract

Let $(H_s(n))_{n \geq 1}$ be an s -dimensional Halton's sequence. Let D_N be the discrepancy of the sequence $(H_s(n))_{n=1}^N$. It is known that $ND_N = O(\ln^s N)$ as $N \rightarrow \infty$. In this paper we prove that this estimate is exact:

$$\overline{\lim}_{N \rightarrow \infty} N \ln^{-s}(N) D_N > 0.$$

To cite this article: A. Nom1, C. R. Acad. Sci. Paris, Ser. I .

Résumé

Sur la limite inférieure de la discrédance de la séquence de Halton.

Soit $(H_s(n))_{n \geq 1}$ une séquence de Halton s -dimensions. Soit D_N la discrédance de la séquence $(H_s(n))_{n=1}^N$. Il est connu que $ND_N = O(\ln^s N)$ que $N \rightarrow \infty$. Dans cet article, nous montrons que cette estimation est exacte :

$$\overline{\lim}_{N \rightarrow \infty} N \ln^{-s}(N) D_N > 0.$$

Pour citer cet article : A. Nom1, C. R. Acad. Sci. Paris, Ser. I .

1. Introduction

Let $(\beta_n)_{n \geq 1}$ be a sequence in the unit cube $[0, 1]^s$, $\mathbf{y} = (y_1, \dots, y_s)$, $B(\mathbf{y}) = [0, y_1) \times \dots \times [0, y_s) \subseteq [0, 1]^s$,

$$\Delta(B(\mathbf{y}), (\beta_n)_{n=1}^N) = \sum_{1 \leq n \leq N} (\mathbf{1}_{B(\mathbf{y})}(\beta_n) - y_1 \cdots y_s), \quad \text{where } \mathbf{1}_{B(\mathbf{y})}(\mathbf{x}) = 1, \text{ if } \mathbf{x} \in B(\mathbf{y}), \quad (1)$$

Email address: mlevin@math.biu.ac.il (Mordechay B. Levin).

and $\mathbf{1}_{B(\mathbf{y})}(\mathbf{x}) = 0$, if $\mathbf{x} \notin B(\mathbf{y})$. We define the star *discrepancy* of a N -point set $(\beta_n)_{n=1}^N$ as

$$D^*((\beta_n)_{n=1}^N) = \sup_{0 < y_1, \dots, y_s \leq 1} \left| \frac{1}{N} \Delta(B(\mathbf{y}), (\beta_n)_{n=1}^N) \right|. \quad (2)$$

Let $(\beta_n)_{n \geq 1}$ be an arbitrary sequence in $[0, 1]^s$. In 1954, Roth proved that

$$\limsup_{N \rightarrow \infty} N(\ln N)^{-\frac{s}{2}} D^*((\beta_n)_{n=1}^N) > 0.$$

According to the well known conjecture (see, e.g., [1, p.283]), this estimate can be improved

$$\limsup_{N \rightarrow \infty} N(\ln N)^{-s} D^*((\beta_n)_{n=1}^N) > 0. \quad (3)$$

In 1972, W. Schmidt proved this conjecture for $s = 1$. For $s = 2$, Faure and Chaix [4] proved (3) for a class of (t, s) -sequences. For a review of research on this conjecture, see for example [2].

Definition. An s -dimensional sequence $(\beta_n)_{n \geq 1}$ is of *low discrepancy* (abbreviated *l.d.s.*) if $D^*((\beta_n)_{n=1}^N) = O(N^{-1}(\ln N)^s)$ for $N \rightarrow \infty$.

Let $p \geq 2$ be an integer,

$$n = \sum_{i \geq 0} e_{p,i}(n) p^i, \text{ with } e_{p,i}(n) \in \{0, 1, \dots, p-1\}, \quad \text{and} \quad \phi_p(n) = \sum_{i \geq 0} e_{p,i}(n) p^{-i-1}. \quad (4)$$

Van der Corput (see [3, ref. 1891]) proved that $(\phi_p(n))_{n \geq 0}$ is a 1-dimensional l.d.s. Let

$$H_s(n) = (\phi_{p_1}(n), \dots, \phi_{p_s}(n)), \quad n = 0, 1, 2, \dots, \quad (5)$$

where $p_1, \dots, p_s \geq 2$ are pairwise coprime integers. Halton (see [3, ref. 729]) proved that $(H_s(n))_{n \geq 0}$ is an s -dimensional l.d.s. For other examples of l.d.s. see e.g. [1], [3]. In §2 we will prove

Theorem. Let $p_0 = p_1 p_2 \cdots p_s$, $s \geq 2$ and $m_0 = [2p_0 \log_2 p_0] + 2$. Then

$$\sup_{1 \leq N \leq 2^{m_0}} N D^*((H_s(n))_{n=1}^N) \geq m^s (8p_0)^{-1} \quad \text{for } m \geq p_0.$$

Remark. This result supports the conjecture (3). In [5], we received a similar result for sequences obtained from algebraic lattices. In [6], we proved a similar result for some (t, s) -sequences (see also [7]).

2. Proof of the Theorem.

Let $x_i = \sum_{j \geq 1} x_{i,j} p_i^{-j}$, with $x_{i,j} \in \{0, 1, \dots, p_i - 1\}$, $[x_i]_r = \sum_{1 \leq j \leq r} x_{i,j} p_i^{-j}$, $i = 1, \dots, s$, $r = 1, 2, \dots$. By (4), we have $\phi_{p_i}(n) \in [[x_i]_r, [x_i]_r + p_i^{-r})$ if and only if $n \equiv \dot{x}_{i,r} \pmod{p_i^r}$, where $\dot{x}_{i,r} = \sum_{1 \leq j \leq r} x_{i,j} p_i^{j-1}$. Let $\mathbf{r} = (r_1, \dots, r_s)$, $P_{\mathbf{r}} = p_1^{r_1} \cdots p_s^{r_s}$ and $M_{i,\mathbf{r}} \equiv (P_{\mathbf{r}} p_i^{-r_i})^{-1} \pmod{p_i^{r_i}}$. Using the Chinese Remainder Theorem, we get

$$\phi_{p_i}(n) \in [[x_i]_{r_i}, [x_i]_{r_i} + p_i^{-r_i}), \text{ for } i = 1, \dots, s \iff n \equiv \ddot{x}_{\mathbf{r}} \pmod{P_{\mathbf{r}}} \text{ with } \ddot{x}_{\mathbf{r}} = \sum_{i=1}^s M_{i,\mathbf{r}} P_{\mathbf{r}} p_i^{-r_i} \dot{x}_{i,r_i}. \quad (6)$$

It is easy to verify that if $r'_i \geq r_i$, for all $i = 1, \dots, s$, then

$$\ddot{x}_{\mathbf{r}'} \equiv \ddot{x}_{\mathbf{r}} \pmod{P_{\mathbf{r}}}. \quad (7)$$

We consider the case $x_{i,r_i} \neq 0$ for all $i = 1, \dots, s$. We obtain from (6) that

$$\phi_{p_i}(n) \in [[x_i]_{r_i} - p_i^{-r_i}, [x_i]_{r_i}), \text{ for } i = 1, \dots, s \iff n \equiv \ddot{x}_{\mathbf{r}} - \sum_{i=1}^s M_{i,\mathbf{r}} P_{\mathbf{r}} p_i^{-1} \pmod{P_{\mathbf{r}}}. \quad (8)$$

Let $p_0 = p_1 p_2 \cdots p_s$, $\check{p}_i = p_0/p_i$, $\tau_i = \min\{1 \leq k < \check{p}_i | p_i^k \equiv 1 \pmod{\check{p}_i}\}$, $i = 1, \dots, s$. Let $\mathbf{y} = (y_1, \dots, y_s)$ with $y_i = \sum_{1 \leq j \leq m} p_i^{-j\tau_i}$, $[y_i]_{\tau_i k_i} = \sum_{1 \leq j \leq k_i} p_i^{-j\tau_i}$, and let $\dot{y}_{i, \tau_i k_i} = \sum_{1 \leq j \leq k_i} p_i^{j\tau_i-1}$, $k_i \geq 1$, $i = 1, \dots, s$, $\mathbf{k} = (k_1, \dots, k_s)$, $B(\mathbf{y}) = [0, y_1) \times \cdots \times [0, y_s) \subset [0, 1)^s$, $B_{\mathbf{k}} = \prod_{1 \leq i \leq s} [[y_i]_{\tau_i k_i} - p_i^{-k_i \tau_i}, [y_i]_{\tau_i k_i})$, $\boldsymbol{\tau} = (\tau_1, \dots, \tau_s)$, $u \cdot v = (u_1 v_1, \dots, u_s v_s)$. We have

$$B(\mathbf{y}) = \bigcup_{1 \leq k_1, \dots, k_s \leq m} B_{\mathbf{k}}, \quad \text{and} \quad \mathbf{1}_{B(\mathbf{y})}(\mathbf{z}) - y_1 \cdots y_s = \sum_{1 \leq k_1, \dots, k_s \leq m} (\mathbf{1}_{B_{\mathbf{k}}}(\mathbf{z}) - P_{\boldsymbol{\tau} \cdot \mathbf{k}}^{-1}). \quad (9)$$

Let $\hat{y}_{\boldsymbol{\tau} \cdot \mathbf{k}} = \sum_{i=1}^s M_{i, \boldsymbol{\tau} \cdot \mathbf{k}}, P_{\boldsymbol{\tau} \cdot \mathbf{k}} p_i^{-\tau_i k_i} \dot{y}_{i, \tau_i(k_i-1)}$ and

$$A_{\mathbf{k}} \equiv - \sum_{i=1}^s M_{i, \boldsymbol{\tau} \cdot \mathbf{k}} P_{\boldsymbol{\tau} \cdot \mathbf{k}} p_i^{-1} \pmod{P_{\boldsymbol{\tau} \cdot \mathbf{k}}}, \quad \text{with} \quad A_{\mathbf{k}} \in [0, P_{\boldsymbol{\tau} \cdot \mathbf{k}}). \quad (10)$$

From (6) we get $\ddot{y}_{\boldsymbol{\tau} \cdot \mathbf{k}} = \sum_{i=1}^s M_{i, \boldsymbol{\tau} \cdot \mathbf{k}}, P_{\boldsymbol{\tau} \cdot \mathbf{k}} p_i^{-\tau_i k_i} \dot{y}_{i, \tau_i k_i} \equiv \hat{y}_{\boldsymbol{\tau} \cdot \mathbf{k}} - A_{\mathbf{k}} \pmod{P_{\boldsymbol{\tau} \cdot \mathbf{k}}}$. By (5) and (8), we obtain

$$H_s(n) \in B_{\mathbf{k}} \iff \phi_{p_i}(n) \in [\dot{y}_{i, \tau_i k_i} - p_i^{-\tau_i k_i}, \dot{y}_{i, \tau_i k_i}), \quad \text{for} \quad i = 1, \dots, s \iff n \equiv \hat{y}_{\boldsymbol{\tau} \cdot \mathbf{k}} \pmod{P_{\boldsymbol{\tau} \cdot \mathbf{k}}}. \quad (11)$$

Let

$$\tilde{y}_m := \hat{y}_{\boldsymbol{\tau}(m+1)} \pmod{P_{\boldsymbol{\tau}(m+1)}}, \quad \text{with} \quad \tilde{y}_m \in [0, P_{\boldsymbol{\tau}(m+1)}), \quad \text{where} \quad \boldsymbol{\tau}(m+1) = (\tau_1(m+1), \dots, \tau_s(m+1)). \quad (12)$$

Using (7), we get $\hat{y}_{\boldsymbol{\tau} \cdot \mathbf{k}} - A_{\mathbf{k}} \equiv \ddot{y}_{\boldsymbol{\tau} \cdot \mathbf{k}} \equiv \ddot{y}_{\boldsymbol{\tau}(m+1)} \equiv \hat{y}_{\boldsymbol{\tau}(m+1)} - A_{\boldsymbol{\tau}(m+1)} \equiv \tilde{y}_m \pmod{P_{\boldsymbol{\tau} \cdot \mathbf{k}}}$, with $k_1, \dots, k_s \in [1, m]$. Applying (11), we have

$$H_s(n) \in B_{\mathbf{k}} \iff n \equiv \tilde{y}_m + A_{\mathbf{k}} \pmod{P_{\boldsymbol{\tau} \cdot \mathbf{k}}}.$$

By (11), we get

$$\begin{aligned} & \sum_{n=\tilde{y}_m+N_1 P_{\boldsymbol{\tau} \cdot \mathbf{k}}}^{\tilde{y}_m+(N_1+1)P_{\boldsymbol{\tau} \cdot \mathbf{k}}-1} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau} \cdot \mathbf{k}}^{-1}) = 0, \quad \text{and} \quad \sum_{n=\tilde{y}_m+N_1 P_{\boldsymbol{\tau} \cdot \mathbf{k}}}^{\tilde{y}_m+N_1 P_{\boldsymbol{\tau} \cdot \mathbf{k}}+N_2-1} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau} \cdot \mathbf{k}}^{-1}) \\ &= \sum_{n \in [\tilde{y}_m, \tilde{y}_m+N_2)} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau} \cdot \mathbf{k}}^{-1}) = \sum_{\substack{n \in [\tilde{y}_m, \tilde{y}_m+N_2) \\ n=\tilde{y}_m+A_{\mathbf{k}}}} 1 - N_2 P_{\boldsymbol{\tau} \cdot \mathbf{k}}^{-1} = \mathbf{1}_{[0, N_2)}(A_{\mathbf{k}}) - N_2 P_{\boldsymbol{\tau} \cdot \mathbf{k}}^{-1}, \end{aligned} \quad (13)$$

with $N_1 \geq 0$ and $N_2 \in [0, P_{\boldsymbol{\tau} \cdot \mathbf{k}})$, $N_1, N_2 \in \mathbb{Z}$. From (1) and (9), we get

$$\begin{aligned} \Delta(B(\mathbf{y}), (H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1}) &= \sum_{y_0, m \leq n < y_0, m+N} (\mathbf{1}_{B(\mathbf{y})}(H_s(n)) - y_1 \cdots y_s) \\ &= \sum_{1 \leq k_1, \dots, k_s \leq m} \rho(\mathbf{k}, N), \quad \text{with} \quad \rho(\mathbf{k}, N) = \sum_{y_0, m \leq n < y_0, m+N} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau} \cdot \mathbf{k}}^{-1}). \end{aligned} \quad (14)$$

Let

$$\alpha_m := \frac{1}{P_{\boldsymbol{\tau} m}} \sum_{N=1}^{P_{\boldsymbol{\tau} m}} \Delta(B(\mathbf{y}), (H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1}) = \sum_{1 \leq k_1, \dots, k_s \leq m} \alpha_{m, \mathbf{k}}, \quad \text{with} \quad \alpha_{m, \mathbf{k}} = \frac{1}{P_{\boldsymbol{\tau} m}} \sum_{N=1}^{P_{\boldsymbol{\tau} m}} \rho(\mathbf{k}, N). \quad (15)$$

Bearing in mind (13) and (14), we derive

$$\begin{aligned} \alpha_{m, \mathbf{k}} &= \frac{1}{P_{\boldsymbol{\tau} m}} \sum_{N_1=0}^{P_{\boldsymbol{\tau} m}/P_{\boldsymbol{\tau} \cdot \mathbf{k}}-1} \sum_{N_2=1}^{P_{\boldsymbol{\tau} \cdot \mathbf{k}}} \left(\sum_{n=\tilde{y}_m}^{\tilde{y}_m+N_1 P_{\boldsymbol{\tau} \cdot \mathbf{k}}-1} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau} \cdot \mathbf{k}}^{-1}) \right. \\ &+ \left. \sum_{n=\tilde{y}_m+N_1 P_{\boldsymbol{\tau} \cdot \mathbf{k}}}^{\tilde{y}_m+N_1 P_{\boldsymbol{\tau} \cdot \mathbf{k}}+N_2-1} (\mathbf{1}_{B_{\mathbf{k}}}(H_s(n)) - P_{\boldsymbol{\tau} \cdot \mathbf{k}}^{-1}) \right) = \frac{1}{P_{\boldsymbol{\tau} m}} \sum_{N_1=0}^{P_{\boldsymbol{\tau} m}/P_{\boldsymbol{\tau} \cdot \mathbf{k}}-1} \sum_{N_2=1}^{P_{\boldsymbol{\tau} \cdot \mathbf{k}}} \left(\mathbf{1}_{[0, N_2)}(A_{\mathbf{k}}) - N_2 P_{\boldsymbol{\tau} \cdot \mathbf{k}}^{-1} \right) \end{aligned}$$

$$= \frac{1}{P_{\tau \cdot \mathbf{k}}} \sum_{N_2=1}^{P_{\tau \cdot \mathbf{k}}} \left(\mathbf{1}_{[0, N_2)}(A_{\mathbf{k}}) - N_2 P_{\tau \cdot \mathbf{k}}^{-1} \right) = \frac{P_{\tau \cdot \mathbf{k}} - A_{\mathbf{k}}}{P_{\tau \cdot \mathbf{k}}} - \frac{P_{\tau \cdot \mathbf{k}}(P_{\tau \cdot \mathbf{k}} + 1)}{2P_{\tau \cdot \mathbf{k}}^2} = \frac{1}{2} - \frac{A_{\mathbf{k}}}{P_{\tau \cdot \mathbf{k}}} - \frac{1}{2P_{\tau \cdot \mathbf{k}}}.$$

Using (15), we have

$$\alpha_m = \sum_{1 \leq k_1, \dots, k_s \leq m} \left(\frac{1}{2} - \frac{A_{\mathbf{k}}}{P_{\tau \cdot \mathbf{k}}} - \frac{1}{2P_{\tau \cdot \mathbf{k}}} \right). \quad (16)$$

Taking into account that $M_{i, \tau \cdot \mathbf{k}} \equiv (P_{\tau \cdot \mathbf{k}} p_i^{-\tau_i k_i})^{-1} \equiv \prod_{1 \leq j \leq s, j \neq i} p_j^{-\tau_j k_j} \pmod{p_i^{\tau_i k_i}}$, and that $p_j^{\tau_j} \equiv 1 \pmod{p_i}$ ($i \neq j$), we obtain $M_{i, \tau \cdot \mathbf{k}} \equiv 1 \pmod{p_i}$, $i = 1, \dots, s$. From (10), we get

$$[0, 1) \ni \frac{A_{\mathbf{k}}}{P_{\tau \cdot \mathbf{k}}} \equiv - \sum_{1 \leq i \leq s} M_{i, \tau \cdot \mathbf{k}} P_{\tau \cdot \mathbf{k}} p_i^{-1} / P_{\tau \cdot \mathbf{k}} \equiv -\frac{1}{p_1} - \dots - \frac{1}{p_s} \pmod{1}.$$

Applying (16), we derive

$$\alpha_m = m^s \left(\frac{1}{2} - \{-\beta\} \right) - \sum_{1 \leq k_1, \dots, k_s \leq m} \frac{1}{2P_{\tau \cdot \mathbf{k}}}, \quad \text{with } \beta = \frac{1}{p_1} + \dots + \frac{1}{p_s}, \quad (17)$$

where $\{x\}$ is the fractional part of x . Let $\beta \equiv 1/2 \pmod{1}$. Hence $p_0 = p_1 p_2 \dots p_s \equiv 0 \pmod{2}$. Let $p_\nu \equiv 0 \pmod{2}$ for some $\nu \in [1, s]$. Then

$$b_1 := p_0(p_\nu/2-1)/p_\nu \equiv p_0(\beta-1/p_\nu) = p_0 \sum_{1 \leq i \leq s, i \neq \nu} 1/p_i \pmod{1} \text{ and } b_1 \equiv b_2 \pmod{p_0}, \text{ with } b_2 = \sum_{i \neq \nu} p_0/p_i.$$

Let $j \in [1, s]$ and $j \neq \nu$. We see that $b_1 \equiv 0 \pmod{p_j}$ and $b_2 \not\equiv 0 \pmod{p_j}$. We get a contradiction. Hence $\beta \not\equiv 1/2 \pmod{1}$. We have

$$0 \neq \left| \frac{1}{2} - \{-\beta\} \right| = \left| \frac{1}{2} - \left\{ -\left(\frac{1}{p_1} + \dots + \frac{1}{p_s} \right) \right\} \right| = \frac{|a|}{2p_0}, \quad \text{with some integer } a.$$

Thus $|1/2 - \{-\beta\}| \geq 1/(2p_0)$. Bearing in mind that $P_{\tau \cdot \mathbf{k}} \geq 2^{k_1 + k_2 + \dots + k_s}$, we obtain from (17)

$$|\alpha_m| \geq \frac{m^s}{2p_0} - \frac{1}{2} = \frac{m^s}{2p_0} \left(1 - \frac{p_0}{m^s} \right) \geq \frac{m^s}{4p_0} \quad \text{for } m \geq p_0 > 4. \quad (18)$$

It is easy to see that $\tau_i \leq p_0$, $i = 1, \dots, s$, and $2P_{\tau(m+1)} = 2p_1^{\tau_1(m+1)} \dots p_s^{\tau_s(m+1)} \leq 2^{1+p_0(m+1)\log_2 p_0} \leq 2^{m(1+2p_0\log_2 p_0)} \leq 2^{mm_0}$ with $m_0 = [2p_0\log_2 p_0] + 2$. Using (12), we have that $\tilde{y}_m + P_{\tau m} < 2P_{\tau(m+1)} \leq 2^{mm_0}$. By (18), (15) and (2), we get

$$\begin{aligned} m^s(4p_0)^{-1} &\leq |\alpha_m| \leq \sup_{1 \leq N \leq P_{\tau m}} ND^*((H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1}) \\ &\leq \sup_{1 \leq L, L+N \leq 2P_{\tau(m+1)}} ND^*((H_s(n))_{n=L}^{L+N-1}) \leq 2 \sup_{1 \leq N \leq 2^{mm_0}} ND^*((H_s(n))_{n=1}^N) \quad \text{for } m \geq p_0. \end{aligned}$$

Hence, the Theorem is proved.

References

- [1] Beck, J., Chen, W. W. L., Irregularities of Distribution, Cambridge Univ. Press, Cambridge, 1987.
- [2] Bilyk, D., On Roth's orthogonal function method in discrepancy theory. Unif. Distrib. Theory 6 (2011), no. 1, 143-184.
- [3] Drmota, M., Tichy, R., Sequences, Discrepancies and Applications, Lecture Notes in Mathematics 1651, 1997.
- [4] Faure, H., Chaix, H., Minoration de discrpance en dimension deux, Acta Arith. 76 (1996), no. 2, 149-164.
- [5] Levin M.B., On the lower bound in the lattice point remainder problem for a parallelepiped, <http://arxiv.org/abs/1307.2080>.
- [6] Levin M.B., On the lower bound of the discrepancy of (t, s) sequence, arxiv.org, January 2015.
- [7] Levin M.B., On the lower bound of the discrepancy of Halton's sequence, II, arxiv.org, January 2015.